MATH 312
Lecture 19

DIAGONALIZATION

Last Time:
- Wanted to compute A^k for some $n \times n$ matrix A and huge k...
- Noted that if A is diagonal, i.e.: if $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, then A^k is easy!
- So, wanted $A = SDS^{-1}$ where S is some invertible matrix and D is a diagonal matrix! Now,

$$A^k = SD^kS^{-1} \leftarrow \text{easy!}$$

Since any diagonal matrix $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ acts like this on the basis vectors:

$$De_j = \lambda_j e_j,$$

"eigenvector" "eigenvalue"

we want to solve the "Eigenvalue equation"

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

(for $\vec{x}$ and λ, given A)

PROBLEM: The right side is NON LINEAR if we don't know λ already

Q How to figure out λ without knowing $\vec{x}$?

Ans DETERMINANTS!

If $A\vec{x} = \lambda\vec{x}$,

then $A\vec{x} - \lambda\vec{x} = \vec{0}$

So $(A - \lambda I)\vec{x} = \vec{0}$ (here $I = n \times n$ identity)

So $\vec{x}$ lies in the nullspace of $(A - \lambda I)$.

CLEARLY | we want $\vec{x} \neq \vec{0}$ (otherwise all of this is TRIVIAL: $\vec{0}$ is NEVER an eigenvector!)

So $(A - \lambda I)$ must have a nullspace different from (i.e., LARGER than) just $\vec{0}$.

So $(A - \lambda I)$ must NOT be invertible!

So $\det(A - \lambda I) = 0$.

Def | An EIGENVALUE of an $n \times n$ matrix A is ANY λ for which $\det(A - \lambda I) = 0$

Once we know λ, figuring out its eigenvectors is just computing the nullspace of $(A - \lambda I)$!

Eg 1 $\quad A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$;

STEP 1: Find eigenvalues by solving for λ in $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$$

This step is HARD!! might need to do $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ stuff

So $\lambda = 4$ and $\lambda = -1$ are the eigenvalues of A

STEP 2: For each eigenvalue λ, find an nonzero vector in $N(A - \lambda I)$

a) $\lambda = 4$, $\quad A - 4I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -3x + 3y &= 0 \\ 2x - 2y &= 0 \end{aligned}\right\} \text{dependent equations}$$

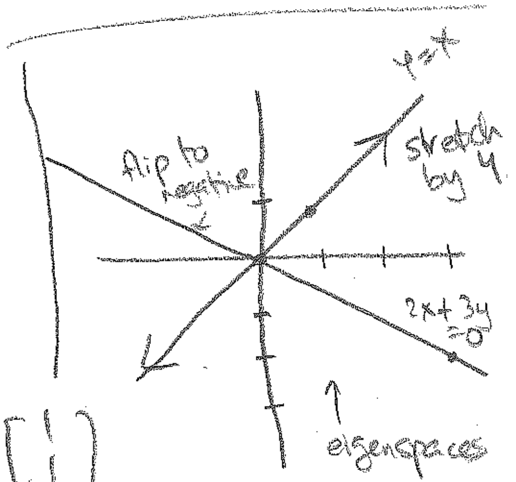
So $x = y$ for all vectors in $N(A - 4I)$, eg $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

b) $\lambda = -1$, $A + I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} 2x + 3y = 0 \\ 2x + 3y = 0 \end{array}\right\} \text{again, a dependent system}$$

So, $2x = -3y$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $N(A+I)$.

eg: $\boxed{\begin{bmatrix} 3 \\ -2 \end{bmatrix}}$



FINAL ANSWER

A has eigenvalues $\lambda_1 = 4$, $\lambda_2 = -1$

An eigenvector for λ_1 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

" " " λ_2 is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Wait, what does this have to do with the promised diagonalization??

Important Theorem: If the eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are independent, then

$$A = SDS^{-1}, \text{ where}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ and } S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

Proof Easier to check that $AS = SD$

$$AS = A \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

But $Av_j = \lambda_j v_j$, so

$$AS = \begin{bmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= SD$$

"ISSUES":

1. Eigenvalues may be repeated!

   $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\lambda_1 = 1$ and $\lambda_2 = 1$.

   But $(A - I)$ has rank 0, so there are still two independent eigenvectors to be found.

2. Eigenvalues may be repeated... BADLY.

   $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$ and $\lambda_2 = 1$.

   But $(A - I)$ has rank 1, so only one indep. eigenvector, so <u>NOT</u> diagonalizable

3. Eigenvalues may be COMPLEX!

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \quad \lambda^2 + 1 = 0$$

$$\Rightarrow \quad \lambda_1 = i \quad \text{and} \quad \lambda_2 = -i.$$

$\underline{\underline{\lambda_1}}$ Now, $N(A - iI)$ is given by solutions to

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $\left.\begin{aligned} -ix - y &= 0 \\ x - iy &= 0 \end{aligned}\right\}$ These are dependent too! Scaling the first one by i gives the second.

So an eigenvector is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

$\underline{\underline{\lambda_2}}$ $N(A + iI)$ similarly contains $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

So, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = S \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} S^{-1}$

where $S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, so

$$S^{-1} = \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \text{ etc.}$$